THE SAMPLE SPACE

One of many ways to partition the set of all possible outcomes

Egan J. Chernoff & Gale L. Russell
University of Saskatchewan, Canada
<egan.chernoff@usask.ca>
<gale.russell@usask.ca>

In this article, we discuss how acknowledging and embracing that the sample space is one of many ways to partition the set of all possible outcomes impacts the teaching and learning of sample space and probability. After recounting an exchange surrounding two viable answers to a probability question, we detail how developments arising from mathematics education research investigating the partitioning of all possible outcomes can be integrated into the mathematics classroom. As a result, we present a unique perspective to normatively incorrect responses.

With the recent publication of *Focus in High School Mathematics: Reason and Sense Making in Probability and Statistics* (National Council of Teachers of Mathematics (NCTM), 2009), I decided to revisit a memorable exchange that occurred while teaching a lesson on probability in my former high school classroom. The purpose of this article is to recount my own reasoning and sense making of that exchange and, in doing so, present a novel approach to aid in teaching sample space and probability.

The sample space

According to the NCTM's *Principles and Standards for School Mathematics for Grades 9–12*, students should "understand the concepts of sample space and probability distribution and construct sample spaces and distributions in simple cases" (NCTM, 2000, p. 324). However, as documented in the probability chapter of the *Second Handbook of Research on Mathematics Teaching and Learning* (Lester, 2007), "although the concept of sample space appears to be a relatively straight forward aspect of the mathematics of random phenomena, it is more subtle and elusive than it appears" (Jones, Langrall, & Mooney, 2007, p. 920). Subtle? Elusive? According to the glossary of the *Australian Curriculum: Mathematics* (2009a) and other popular sources (e.g., InterMath, WolframMathWorld, Connecting

Mathematics, The Math Forum) the *sample space* is the *set of all possible outcomes of a chance experiment*. All you do is list all possible outcomes! Was I missing something? As you are about to see, in a re-creation of the exchange between a former student (who, for the purposes of this article, I will call Dexter) and me, something was definitely missing: my understanding that the sample space is *but one of many ways* to partition the set of all possible outcomes.

The question

The exchange, which occurred between Dexter and myself, was focused on one of my all-time favourite probability questions, which relates to content description ACSMP246 from Year 9 of the *Australian Curriculum: Mathematics* (2009b):

> A fair coin is flipped three times.
> What is the probability of two heads and a tail?

My affinity for this question is based on the notion that a mathematics classroom will, more often than not, come up with two "viable" answers: $\frac{1}{4}$ and $\frac{3}{8}$. On the one hand, those individuals who answer $\frac{1}{4}$ demonstrate that there are 4 possible outcomes (using H for head and T for tail: 3H and 0T, 2H and 1T, 1H and 2T, and 0H and 3T) of which 1 outcome (2H and 1T) is favourable and, as such, the probability is $\frac{1}{4}$. On the other hand, those individuals who answer, correctly, demonstrate that there are 8 possible outcomes (HHH, HHT, HTH, THH, HTT, THT, TTH, and TTT) of which 3 outcomes (HHT, HTH, and THH) are favourable and, as such, the probability is $\frac{3}{8}$. Discussing these two answers has, in the past, provided us with an opportunity to address many of the sample space concepts found in the Statistics and Probability (Chance) content for Years 7, 8 and 9 of the *Australian Curriculum: Mathematics* (2009a) and, further, the NCTM's (2000) *Data Analysis and Probability Standards*. For example, having two answers allowed us to discuss (amongst other concepts): the difference between elementary outcomes and events; the notion of favourable outcomes; the classical interpretation of probability; and when order matters versus when it does not. However, as you will read below, it would take our recounting of Dexter's unique perspective on the concept of "all possible outcomes" for us to see how the question can be used to discuss sample space construction and, further, that the sample space is, indeed, more subtle and elusive than it appears.

The exchange

In what follows, we recreate (part of) the unique exchange between Dexter, who argued that the answer is $\frac{1}{4}$, and me, who argued that the answer is $\frac{3}{8}$. We pick up the conversation having established that there are two possible answers and I am attempting to convince Dexter he has not listed all possible outcomes.

Mr. C.: Dexter, the answer cannot be one quarter because you have not listed all possible outcomes.

Dexter: Actually, that's not true! I have listed all the possible outcomes: three heads and no tails, two heads and one tail, one head and

two tails, and no heads and three tails represents all possible outcomes.

Mr. C.: No Dexter, I said to list all possible outcomes…

Dexter: I have.

Mr. C.: Dexter, why won't you listen? I said, list all possible outcomes…

Dexter: Mr. C., honestly, I don't think you're listening. I have listed all possible outcomes.

Mr. C.: Dexter: do you agree or disagree that the sample space is the set of all possible outcomes?

Dexter: Ok… I agree.

Mr. C.: Good. Then you should also agree, if, as you said, the sample space is the set of all possible outcomes, you haven't listed the sample space…

Dexter: I disagree. If the sample space is the set of all possible outcomes then, in fact, I have listed the sample space (and there are four possible outcomes: 3H and 0T, 2H and 1T, 1H and 2T, 0H and 3T).

Mr. C.: Ok Dexter, I guess we'll have to agree to disagree…

Dexter: Hold on, let me ask you a question: Explain to me how I haven't listed all possible outcomes?

Thinking back to my response to Dexter's request, I was like a deer caught in the headlights. With the rest of the class looking on and waiting, I attempted to reconcile, first for myself and then for the rest of the class, Dexter's comment. Then it dawned on me: He was right; he had listed all possible outcomes for three flips of a fair coin. However, I maintained, at that moment, that although Dexter had listed all the possible outcomes for three flips of a fair coin, he still had incorrectly identified the sample space because, after all, not all of the outcomes he had presented were equally likely to occur. For example, it is three times more likely to obtain 2 heads and 1 tail (HHT, HTH, THH) as opposed to 3 heads and 0 tails (HHH) for three flips of a fair coin. Sure, I could have been more specific about the equiprobability characteristic of the sample space, but, reflecting on Dexter's responses, there seemed to be more going on (that I was missing), which was preventing us from moving forward. After class, I decided to sit down and take stock of what had happened.

The aftermath

For the rest of the day and for the next few days, my thoughts were consumed with Dexter's unconventional listing (i.e., partitioning) of all possible outcomes. I started to wonder, what were some the other unconventional listings of all possible outcomes? At my desk, I started rearranging the 8 equally likely outcomes from 3 flips of a fair coin and, sure enough, different, alternative listings of all possible outcomes began to emerge. Dexter's ratio of heads to tails listing of all possible outcomes (3H and 0T, 2H and 1T, 1H and 2T, and 0H and 3T) was accompanied by: the length of a run listing (run of length 1, run of length 2, and run of length 3), the number of switches listing (0 switches, 1 switch, 2 switches), the heads or tails first listing (heads on the first flip, tails on the first flip), and others.

The number of readily alternative listings that emerged led me in two directions. First, I wanted to mathematise the situation; I wanted to determine how many alternative listings of all possible outcomes there were.

Utilising the notion of a Bell number—which is "the number of ways a set of n elements can be partitioned into nonempty subsets" (Weisstein, 2007)—I determined that there were 4140 possible ways to alternatively list all possible outcomes for three flips of a fair coin. Second, given the alternative listings of all possible outcomes I was able to come up with and given the number of possible alternative listings (i.e., 4140), I began to question my strategy of having students list all possible outcomes when answering certain probabilistic questions. After all, if there were 4140 possible ways of listing all possible outcomes and only one of those 4140 lists corresponded to the sample space, it seems much more likely that a student would come up with one of the alternative ways of listing all possible outcomes, like Dexter, before they came up with the sample space. Taking my new ideas back to the classroom, I dedicated an afternoon, albeit with a different class, to having students come up with alternative listings of all possible outcomes for three flips of a fair coin. The results were amazing. Then, out of the blue, another student, Rita, asked, "If none of these are the sample space, what do we call them?" Great question!

Investigating partitions of the set of all possible outcomes

My recent investigations—focusing on the notion that the sample space is but one of many listings of all possible outcomes (e.g., Chernoff, 2009; Chernoff & Zazkis, 2011)—have led to what I consider to be some interesting results. In what follows, we detail two particular, interrelated ideas and demonstrate how they can be utilised to provide a fresh perspective on the teaching and learning of probability in the classroom.

Event description alignment

The process of event description alignment—where responses are aligned with alternative, more appropriate set descriptions of all possible outcomes—has become integral to making sense of what I previously had minimised as incorrect responses. When asked to compare the relative likelihood of two sequences of coin flips, invariably some students declare, for example, that the sequence HHH is less likely to occur than the equally likely sequence HTT and they reason that you are more likely to get a mixture of heads and tails rather than all the same. In the past, my response to an "error" of this type was to simply reiterate that each of the possible outcomes of the sample space is equally likely to occur—as I did with Dexter. However, now, I try to compare their response with more appropriate set descriptions of all possible outcomes.

For the relative likelihood example, comparing the mixture response to a mixture description of all possible outcomes, that is {mixture of heads and tails, no mixture} or {{HTT, THT, TTH, HHT, HTH, THH}, {HHH, TTT}} is more appropriate for understanding students' reasoning than comparing their response to the sample space, that is, {HHH, HTT, THT, TTH, HHT, HTH, THH, TTT}. The response, previously seen as incorrect, displays correct probabilistic reasoning when declaring that a mixture of heads and tails is more likely—three times—than no mixture. By comparing students' responses to more appropriate set descriptions, we have found that, in fact, students are often demonstrating correct probabilistic reasoning and, in some cases, their reasoning is quite subtle. Continuing with the relative

likelihood example, some individuals declare that HHH is less likely to occur than the equally likely sequence HTT because you are less likely to get three heads and no tails than one head and two tails. In this instance, comparing the response to the set of all possible outcomes organised according to the ratio of heads to tails, that is, {3H and 0T, 2H and 1T, 1H and 2T, 0H and 3T} (or equivalently {{HHH}, {HTT, THT, TTH}, {HHT, HTH, THH}, {TTT}}) is more appropriate than comparing the response to the sample space, that is, {HHH, HTT, THT, TTH, HHT, HTH, THH, TTT}. While, normatively, the student has made an error in declaring one sequence as less likely, more importantly, we are able to see, through event description alignment, correct probabilistic reasoning when declaring that three heads and no tails (i.e., HHH) is less likely than two heads and one tail (i.e., HHT, HTH, THH). However, as Rita asked, "If none of these are the sample space, what do we call them"?

The sample set

To address certain issues associated with the multitude of alternative set descriptions for all possible outcomes, the notion of a sample set—defined "as any set of all possible outcomes, where the elements of this set do not need to be equiprobable" (Chernoff & Zazkis, 2011, p. 18)—was introduced. Consider one of the examples from above: {mixture of heads and tails, no mixture of heads and tails} (or equivalently {{HTT, THT, TTH, HHT, HTH, THH}, {HHH, TTT}}) would be a sample set for three flips of a fair coin because all possible outcomes are listed. Of note, in this case, the two elements of the set—mixture of heads and tails and no mixture—are not equiprobable. Further, {3H and 0T, 2H and 1T, 1H and 2T, 0H and 3T} (or equivalently {{HHH}, {HTT, THT, TTH}, {HHT, HTH, THH}, {TTT}}) is also a sample set for three flips of a fair coin because all possible outcomes are listed. "In general, a sample set is a set of all possible outcomes, that is to say the sample space, which is not listed in conventional form" (Chernoff & Zazkis, 2011, p. 19). Finally, I am able to answer Rita's question: we call "them" (i.e., alternative listings or partitions of all possible outcomes) sample sets. As with event description alignment, the sample set has played a crucial role in making sense of a certain individual's probabilistic reasoning and, as a result, the sample set now plays a vital role in Chernoff and Zazkis' (2011) alternative approach to assigning probabilities.

Revisiting the question that was at the centre of the exchange between Dexter and me (i.e., "A fair coin is flipped three times. What is the probability of two heads and a tail?"), the sample set would have allowed us to overcome our impasse and eventually, and alternatively, get to the same response of $\frac{3}{8}$. Within the terminology introduced above, both Dexter and I had, in fact, listed all possible outcomes for three flips of a fair coin; however, I had listed all the possible outcomes as the sample space {HHH, HTT, THT, TTH, HHT, HTH, THH, TTT} and Dexter had listed all the possible outcomes as one sample set {3H, 1H & 2T, 2H & 1T, 3T} (or equivalently {{HHH}, {HTT, THT, TTH}, {HHT, HTH, THH}, {TTT}}). While treating the outcomes of the sample set as outcomes of the sample space (i.e., equiprobable) would lead to an incorrect answer of $\frac{1}{4}$, it is still possible to build upon Dexter's response, and get to the answer of $\frac{3}{8}$, especially if we take into account that outcomes of the sample set do not need to be equally likely.

Conclusion

In the past, when students provided a list of outcomes for a probability question that was not the sample space, I told them unequivocally that their list was incomplete. It took Dexter's persistence and Rita's questioning to help me realise that, in fact, my analysis was incorrect. As detailed above, there a great number of ways to partition the sample space (i.e., a great number of sample sets) and my job is not to deny these variable sample sets that, it appears, are intuitive to the students, but to build upon their ideas to help the students attain and recognize the sample space. Through discussion and different forms of representation, teachers can help students move confidently into the elusive and subtle world of sample spaces and their equiprobable outcomes, and into a more meaningful and useful understanding of probability itself. Perhaps, one day, our mathematics classrooms will be as focused on the *multispace* (i.e., the set of all sample sets) as it is, today, on the sample space.

References

Australian Curriculum, Assessment and Reporting Authority (2009a). *The Australian Curriculum: Mathematics*. Retrieved from: http://www.australiancurriculum.edu.au/Mathematics/Rationale

Australian Curriculum, Assessment and Reporting Authority (2009b). *The Australian Curriculum: Mathematics*. Retrieved from: http://www.australiancurriculum.edu.au/Elements/76197c32-469d-42c9-a443-9e4600a2536c

Chernoff, E. J. (2009). Sample space partitions: An investigative lens. *Journal of Mathematical Behavior, 28*(1), 19–29.

Chernoff, E. J., Zazkis, R. (2011). From personal to conventional probabilities: From sample set to sample space. *Educational Studies in Mathematics 77*(1), 15–33.

Jones, G. A., Langrall, C. W., & Mooney, E. S. (2007). Research in probability: Responding to classroom realties. In F. K. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 909–955). New York: Macmillan.

Lester, F. K. (2007). *Second handbook of research on mathematics teaching and learning*. New York: Macmillan.

National Council of Teachers of Mathematics (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: NCTM.

National Council of Teachers of Mathematics (2000). *Principles and standards for school Mathematics*. Reston, VA: NCTM.

National Council of Teachers of Mathematics (2009). *Focus in high school mathematics: Reason and sense making in probability and statistics*. Reston, VA: NCTM.

Shaughnessy, M. J., Chance, B., & Kranendonk, H. (2010). *Focus in high school mathematics: Reasoning and sense making in statistics and probability*. Reston, VA: NCTM.

Weisstein, E. W. (2007). *Bell number*. Retrieved from: http://mathworld.wolfram.com/BellNumber.html